Enriching algebras over coalgebras
and operads over cooperads

M. Anel

ETH Zürich
matthieu.anel@math.ethz.ch

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This is a work in progress with A. Joyal.

We are trying to understand Koszul duality from a conceptual point of view.

We still don’t understand Koszul duality, but we discovered some category theory underlying the bar and cobar constructions.
Main theorem

Let \((\mathbf{V}, \otimes, 1, [-,-])\) be a symmetric monoidal closed locally presentable category and let \(P\) be a cocommutative Hopf colored operad in \(\mathbf{V}\).

Theorem (A-J)

1. The category \(P\)-Coalg is symmetric monoidal closed.
2. The category \(P\)-Alg is enriched, tensored, cotensored and symmetric monoidal over \(P\)-Coalg.

Corollary

Let \(P = \text{As the associative operad}\).

1. The category Coalg of coassociative coalgebras is symmetric monoidal closed.
2. The category Alg of associative algebras is enriched, tensored, cotensored and symmetric monoidal over Coalg.
Main theorem

Let \((\mathcal{V}, \otimes, 1, [-, -])\) be a symmetric monoidal closed locally presentable category and let \(P\) be a cocommutative Hopf colored operad in \(\mathcal{V}\).

Theorem (A-J)

1. The category \(P\text{-Coalg}\) is symmetric monoidal closed.
2. The category \(P\text{-Alg}\) is enriched, tensored, cotensored and symmetric monoidal over \(P\text{-Coalg}\).
Main theorem

Let $\left( V, \otimes, 1, [-,-] \right)$ be a symmetric monoidal closed locally presentable category and let $P$ be a cocommutative Hopf colored operad in $V$.

Theorem (A-J)

1. The category $P\text{-Coalg}$ is symmetric monoidal closed.
2. The category $P\text{-Alg}$ is enriched, tensored, cotensored and symmetric monoidal over $P\text{-Coalg}$.

Corollary

Let $P = \text{As}$ the associative operad.

1. The category $\text{Coalg}$ of coassociative coalgebras is symmetric monoidal closed.
2. The category $\text{Alg}$ of associative algebras is enriched, tensored, cotensored and symmetric monoidal over $\text{Coalg}$.
Main theorem

Corollary

Let $P = K$ a category (in $\textbf{Set}$).

1. The category of functors $[K^{\text{op}}, V]$ is symmetric monoidal closed.

2. The category of functors $[K, V]$ is enriched, tensored, cotensored and symmetric monoidal over $[K^{\text{op}}, V]$. 
Main theorem

Corollary
Let $P = K$ a category (in $\mathbf{Set}$).

1. The category of functors $[K^{\text{op}}, V]$ is symmetric monoidal closed.
2. The category of functors $[K, V]$ is enriched, tensored, cotensored and symmetric monoidal over $[K^{\text{op}}, V]$.

Corollary
Let $P = \mathcal{OP}$ be the operad of $K$-colored operads.

1. The category $\mathbf{coOp}(K)$ of $K$-colored cooperads is symmetric monoidal closed.
2. The category $\mathbf{Op}(K)$ of $K$-colored operads is enriched, tensored, cotensored and symmetric monoidal over $\mathbf{coOp}(K)$. 
Part I - Hopf operads
Colored operad

Let $K$ be a set (could be a category). We put $S(K)$ for the free symmetric monoidal category on $K$.

Let $(\mathbf{V}, \otimes)$ be a symmetric monoidal category.

A \emph{$K$-colored operad} $P$ in $\mathbf{V}$ is the data of a functor

$$P : S(K)^{op} \times K \to \mathbf{V}$$

which is a monoid for the substitution monoidal structure

$$P \circ P \to P \quad \text{and} \quad I \to P.$$
Colored operad

Concretely, this amounts to the data of

- objects
  \[ P^k_k = P^{k_1, \ldots, k_n}_k \in \mathbf{V} \]
  (where the \( k, k_i \) are in \( K \))

- actions of symmetric groups related to repetition of elements in \( \overline{k} \)

- and maps

\[ P^{k_1, \ldots, k_n}_k \otimes P^{\ell_1}_{k_1} \otimes \cdots \otimes P^{\ell_n}_{k_n} \longrightarrow P^{\ell_1 + \cdots + \ell_n}_{k} \]

\[ 1 \rightarrow P^k_k \]

satisfying associativity and unitality conditions.
Colored operad - examples

- If $P[n]$ is a unisorted operad (Associative, Commutative, Poisson, Lie, $L_\infty$, $A_\infty$...) we put $K = \{\ast\}$ and

  \[
  P^*_\ast, \ldots, \ast := P[n]
  \]
Colored operad - examples

- If $P[n]$ is a unisorted operad (Associative, Commutative, Poisson, Lie, $L_\infty$, $A_\infty$...) we put $K = \{\ast\}$ and

$$P^* \xrightarrow{n \text{ times}} \cdots \ast := P[n]$$

- If $B$ is an associative algebra, we put $K = \{\ast\}$,

$$P^* := B$$

and all $P$s are other 0.
Colored operad - examples

- If $P[n]$ is a unisorted operad (Associative, Commutative, Poisson, Lie, $L_\infty$, $A_\infty$,...) we put $K = \{\ast\}$ and

\[
\underbrace{P_\ast, \ldots, \ast}_{\text{n times}} := P[n]
\]

- If $B$ is an associative algebra, we put $K = \{\ast\}$,

\[
P_\ast := B
\]

and all $Ps$ are other 0.

- If $K$ is a category, we put $K = ob(K)$,

\[
P^k_{k'} := K(k, k')
\]

and all other $Ps$ are 0.
Colored $P$-algebra

For a covariant functor $A : K \to V$ we shall denote the value at $k \in K$ by $A_k$.

If $\overline{k} = (k_1, \ldots, k_n)$ we put $A_{\overline{k}} = A_{k_1} \otimes \cdots \otimes A_{k_n}$.

Let $P$ a $K$-colored operad.
A $P$-algebra is a functor $A : K \to V$ together with maps

$$P_{\overline{k}}^k \otimes A_{\overline{k}} \to A_k$$

satisfying associativity and unitality conditions.
Colored $P$-algebras - examples

- If $P$ is a unisorted operad, an algebra $A$ is a \textit{unisorted $P$-algebra}

\[ P[n] \otimes A^\otimes n \longrightarrow A \]
Colored $P$-algebras - examples

- If $P$ is a unisorted operad, an algebra $A$ is a **unisorted** $P$-algebra

\[ P[n] \otimes A^\otimes n \longrightarrow A \]

- If $P = B$ is an associative algebra, an algebra $A$ is a **left module**

\[ B \otimes A \longrightarrow A \]
Colored $P$-algebras - examples

- If $P$ is a unisorted operad, an algebra $A$ is a unisorted $P$-algebra

\[ P[n] \otimes A^n \longrightarrow A \]

- If $P = B$ is an associative algebra, an algebra $A$ is a left module

\[ B \otimes A \longrightarrow A \]

- If $P = K$ is a category, an algebra $A$ is a covariant functor

\[ K \rightarrow V. \]
Colored $P$-coalgebra

For a contravariant functor $C : K^{op} \rightarrow \mathbf{V}$ we shall denote the value at $k \in K$ by $C^k$

If $\overline{k} = (k_1, \ldots, k_n)$ we put $C^{\overline{k}} = C^{k_1} \otimes \cdots \otimes C^{k_n}$.

Let $P$ a $K$-colored operad.

A $P$-coalgebra is a functor $C : K^{op} \rightarrow \mathbf{V}$ together with maps

$$P_{k}^{\overline{k}} \otimes C^k \rightarrow C^{\overline{k}}$$

satisfying coassociativity and counitality conditions.
Colored $P$-algebras - examples

- If $P$ is a unisorted operad, a coalgebra $C$ is a unisorted $P$-coalgebra

$$P[n] \otimes C \longrightarrow C^\otimes n$$

- If $P = B$ is an associative algebra, a coalgebra $C$ is a right module.

- If $P = K$ is a category, a coalgebra $C$ is a contravariant functor $K^{\text{op}} \to V$. 
Colored $P$-algebras - examples

- If $P$ is a unisorted operad, a coalgebra $C$ is a unisorted $P$-coalgebra
  \[ P[n] \otimes C \longrightarrow C^\otimes n \]
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  \[ B \otimes C \longrightarrow C \]
Colored $P$-algebras - examples

- If $P$ is a unisorted operad, a coalgebra $C$ is a unisorted $P$-coalgebra
  \[ P[n] \otimes C \rightarrow C^\otimes n \]

- If $P = B$ is an associative algebra, a coalgebra $C$ is a right module.
  \[ B \otimes C \rightarrow C \]

- If $P = K$ is a category, a coalgebra $C$ is a contravariant functor $K^{op} \rightarrow V$. 
Hadamard product

If $P$ and $Q$ are two $K$-colored operad their Hadamard product of $P \otimes Q$ is defined by

$$(P \otimes Q)^k_k := P^k_k \otimes Q^k_k$$

This is again an operad:

$$
\left( P^k_k \otimes Q^k_k \right) \otimes \left( P^\ell_1 \otimes Q^\ell_1 \right) \otimes \cdots \otimes \left( P^\ell_n \otimes Q^\ell_n \right)
$$

$$
= \left( P^k_k \otimes P^\ell_1 \otimes \cdots \otimes P^\ell_n \right) \otimes \left( Q^k_k \otimes Q^\ell_1 \otimes \cdots \otimes Q^\ell_n \right)
$$

$$
\longrightarrow P^\ell_1 \oplus \cdots \oplus \ell_n \otimes Q^\ell_1 \oplus \cdots \oplus \ell_n
$$
Hopf operad

The category \( \mathbf{Op}(K) \) of \( K \)-colored operad is symmetric monoidal for the Hadamard product.

A (cocommutative) Hopf operad is an operad which is a cocommutative comonoid for the Hadamard product.

Equivalently, this says that all \( P^k_k \) are cocommutative comonoids and that the compositions an unit maps are coalgebra maps.
Hopf operad

The category $\mathbf{Op}(K)$ of $K$-colored operad is symmetric monoidal for the Hadamard product.

A (cocommutative) Hopf operad is an operad which is a cocommutative comonoid for the Hadamard product.

Equivalently, this says that all $P_k^\bar{k}$ are cocommutative comonoids and that the compositions an unit maps are coalgebra maps.

Examples:

- all operads in $\mathbf{Set}$ (Associative, Commutative, any category, the operad of $K$-colored operads, ...)
- all operads in $\mathbf{Top}$ ($E_n$, John’s Phyl...)
- the Poisson operad
- any cocommutative bialgebra
(co)algebras over Hopf operad

Let \( P \) be a Hopf operad.

If \( A \) and \( B \) are \( P \)-algebras, their Hadamard product \( A \otimes B \) is defined by

\[(A \otimes B)_k := A_k \otimes B_k\]

it is again a \( P \)-algebra.

\[
P_k^k \otimes A_k \otimes B_k \rightarrow P_k^k \otimes P_k^k \otimes A_k \otimes B_k =
\]

\[
P_k^k \otimes A_k \otimes P_k^k \otimes B_k \rightarrow A_k^k \otimes B_k^k = (A \otimes B)^k
\]
(co)algebras over Hopf operad

Let $P$ be a Hopf operad.

If $A$ and $B$ are $P$-algebras, their Hadamard product $A \otimes B$ is defined by

$$(A \otimes B)_k := A_k \otimes B_k$$

it is again a $P$-algebra.

$$P_k^k \otimes A_k \otimes B_k \longrightarrow P_k^k \otimes P_k^k \otimes A_k \otimes B_k =$$

$$P_k^k \otimes A_k \otimes P_k^k \otimes B_k \longrightarrow A^k \otimes B^k = (A \otimes B)^k$$

Similarly, if $C$ and $D$ are $P$-coalgebras, their Hadamard product $C \otimes D$ defined by

$$(C \otimes D)^k := C^k \otimes D^k$$

is again a $P$-coalgebra.
Part II - SWEEDLER THEORY
Sweedler theory

Let $P$ be a colored operad in a symmetric monoidal closed locally presentable category $V$.

Let $\mathbf{P-Alg}$ and $\mathbf{P-Coalg}$ be the categories of $P$-algebras and of $P$-coalgebras.

**Theorem (folklore)**

1. $\mathbf{P-Alg}$ and $\mathbf{P-Coalg}$ are locally presentable.
2. There exists a monadic adjunction $U: \mathbf{P-Alg} \rightarrow V$.
3. There exists a comonadic adjunction $P^\vee: V \rightarrow \mathbf{P-Coalg}$.

$P^\vee$ is not an analytic comonad (cooperad), hence difficult to describe explicitly.
Sweedler theory

Let $P$ be a colored operad in a symmetric monoidal closed locally presentable category $V$.

Let $P$-Alg and $P$-Coalg be the categories of $P$-algebras and of $P$-coalgebras.

**Theorem (folklore)**

1. $P$-Alg and $P$-Coalg are locally presentable.
2. There exists a monadic adjunction

   $$U : \text{P-Alg} \leftrightarrow V^K : P.$$ 

3. There exists a comonadic adjunction

   $$P^\vee : V^K \leftrightarrow P$"Coalg : U.$$

$P^\vee$ is not an analytic comonad (cooperad), hence difficult to describe explicitly.
Sweedler theory of a Hopf operad

Let $P$ be a colored Hopf operad, there exists six functors

- Tensor product $\otimes : P\text{-Coalg} \times P\text{-Coalg} \to P\text{-Coalg}$
- Internal hom $\text{Hom} : P\text{-Coalg}^{\text{op}} \times P\text{-Coalg} \to P\text{-Coalg}$
- Sweedler hom $\{−, −\} : P\text{-Alg}^{\text{op}} \times P\text{-Alg} \to P\text{-Coalg}$
- Sweedler product $\triangleright : P\text{-Coalg} \times P\text{-Alg} \to P\text{-Alg}$
- Convolution $[−, −] : P\text{-Coalg}^{\text{op}} \times P\text{-Alg} \to P\text{-Alg}$
- Tensor product $\otimes : P\text{-Alg} \times P\text{-Alg} \to P\text{-Alg}$

such that

Theorem (A-J)

1. $(P\text{-Coalg}, \otimes, \text{Hom})$ is symmetric monoidal closed.
2. $(P\text{-Alg}, \{−, −\}, \triangleright, [−, −], \otimes)$ is enriched, tensored, cotensored and symmetric monoidal over $\text{Coalg}$. 
Sweedler theory of the associative operad

For $P = \text{As}$ the associative operad, there exists six functors

- **tensor product** $\otimes : \text{Coalg} \times \text{Coalg} \to \text{Coalg}$
- **internal hom** $\text{HOM} : \text{Coalg}^{\text{op}} \times \text{Coalg} \to \text{Coalg}$
- **Sweedler hom** $\{-, -\} : \text{Alg}^{\text{op}} \times \text{Alg} \to \text{Coalg}$
- **Sweedler product** $\triangleright : \text{Coalg} \times \text{Alg} \to \text{Alg}$
- **convolution** $[-, -] : \text{Coalg}^{\text{op}} \times \text{Alg} \to \text{Alg}$
- **tensor product** $\otimes : \text{Alg} \times \text{Alg} \to \text{Alg}$

such that

**Theorem**

**(Porst)** $(\text{Coalg}, \otimes, \text{HOM})$ is symmetric monoidal closed.

**(A-J)** $(\text{Alg}, \{-, -\}, \triangleright, [-, -], \otimes)$ is enriched, tensored, cotensored and symmetric monoidal over $\text{Coalg}$. 
Sweedler theory of the associative operad

If we choose $(V, \otimes) = (\text{Set}, \times)$, then $P-\text{Alg} = \text{Mon}$ and $P-\text{Coalg} = \text{Set}$. and the enrichment is trivial.
Sweedler theory of the associative operad

If we choose \((V, \otimes) = (\text{Set}, \times)\), then \(P\text{-Alg} = \text{Mon}\) and \(P\text{-Coalg} = \text{Set}\). and the enrichment is trivial.

If we choose \((V, \otimes) = (\text{Vect}, \otimes)\), then the enrichment is not trivial.
Sweedler theory of the associative operad

If we choose \((V, \otimes) = (\text{Set}, \times)\), then \(P\text{-Alg} = \text{Mon}\) and \(P\text{-Coalg} = \text{Set}\). and the enrichment is trivial.

If we choose \((V, \otimes) = (\text{Vect}, \otimes)\), then the enrichment is not trivial. 

\(P^\vee = T^\vee\) is the cofree coalgebra functor (much bigger than the tensor coalgebra).

\(\text{Hom}\) and \(\{-, -\}\) do not have a simple presentation but

\[\text{Hom}(C, T^\vee(X)) = T^\vee([C, X])\]

\[\{T(X), A\} = T^\vee([X, A]).\]
An **atom** of a coalgebra $C$ is an element $e$ such that $\Delta(e) = e \otimes e$ and $\epsilon(e = 1)$

A **primitive element** $u$ of $C$ with respect to some atom $e$ is an element $e$ such that $\Delta(u) = u \otimes e + e \otimes u$

**Proposition**

- $\text{atom}(\text{Hom}(C, D)) = \text{hom}(C, D)$
- $\text{prim}_f(\text{Hom}(C, D)) = \text{Coder}_f(C, D)$
- $\text{atom}(\{A, B\}) = \text{hom}(A, B)$
- $\text{prim}_f(\{A, B\}) = \text{Der}_f(A, B)$
Sweedler theory of the associative operad

The operation $[-,-]$ is the convolution algebra.

If $C$ is a coalgebra and $A$ an algebra, $[C, A]$ is an algebra for the product

$$[C, A] \otimes [C, A] \xrightarrow{\text{can}} [C \otimes C, A \otimes A] \xrightarrow{[\Delta, m]} [C, A].$$
The operation \([-,-]\) is the convolution algebra.

If \(C\) is a coalgebra and \(A\) an algebra, \([C,A]\) is an algebra for the product

\[
[C,A] \otimes [C,A] \xrightarrow{\text{can}} [C \otimes C, A \otimes A] \xrightarrow{[\Delta,m]} [C, A].
\]

A map \(C \otimes A \to B\) in \(\mathbf{V}\) is called a measuring if the corresponding map \(A \to [C,B]\) is an algebra map.
Sweedler theory of the associative operad

\(\mu : C \otimes A \to B\) is a measuring iff the following diagram commutes

\[
\begin{array}{ccc}
C \otimes A \otimes A & \xrightarrow{\Delta_{C \otimes A^2}} & C \otimes C \otimes A \otimes A \\
& \downarrow C \otimes m_A & \sim \\
C \otimes A & \xrightarrow{\mu} & B
\end{array}
\]

In terms of elements, this gives the formula in \(B\)

\[
\mu(c, aa') = \sum \mu(c^{(1)}, a)\mu(c^{(2)}, a')
\]

(where \(\Delta(c) = \sum c^{(1)} \otimes c^{(2)}\))
Sweedler theory of the associative operad

The algebra $C \triangleright A$ can be defined as the quotient of $T(C \otimes A)$ given by coequalizing the two sides of

$$C \otimes A \otimes A \xrightarrow{\Delta_C \otimes A^2} C \otimes C \otimes A \otimes A \xrightarrow{\cong} C \otimes A \otimes C \otimes A$$

In particular we have

$$C \triangleright T(X) = T(C \otimes X).$$
Sweedler theory of the associative operad

Let $C$ be a coalgebra and $A, B$ be two algebras, we have bijection between the following sets

- **measurings** $C \otimes A \to B$
- **algebra maps** $A \to [C, B]$
- **algebra maps** $C \triangleright A \to B$
- **coalgebra maps** $C \to \{A, B\}$.
Sweedler theory of the associative operad

Let $C$ be a coalgebra and $A$ an algebra, we deduce three kinds of adjunctions

type I $\quad C \triangleright - : \text{Alg} \xleftrightarrow{\sim} \text{Alg} : [C, -]$

type II $\quad [-, A] : \text{Coalg} \xleftrightarrow{\sim} \text{Alg}^{op} : \{-, A\}$

type III $\quad - \triangleright A : \text{Coalg} \xleftrightarrow{\sim} \text{Alg} : \{A, -\}$
Sweedler theory of the associative operad

Type I adjunctions are quite frequent: if $V = \text{Vect}$

- $E$ finite algebra, $E^* \triangleright -$ is left adjoint to $E \otimes -$,
Sweedler theory of the associative operad

Type I adjunctions are quite frequent: if $V = \text{Vect}$

- $E$ finite algebra, $E^* \triangleright -$ is left adjoint to $E \otimes -$,

- $C = k \oplus k\delta$ with $\Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta$

$[C, A] = A[\epsilon]$ and $C \triangleright A = T_A(\Omega_A)$,
Sweedler theory of the associative operad

Type I adjunctions are quite frequent: if \( V = \text{Vect} \)

- \( E \) finite algebra, \( E^* \rhd - \) is left adjoint to \( E \otimes - \),

- \( C = k \oplus k\delta \) with \( \Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta \)
  \([C, A] = A[\epsilon] \) and \( C \rhd A = T_A(\Omega_A) \),

- \( C = T^c(x) \) (tensor coalgebra)
  \([C, A] = A[t] \) and \( C \rhd A = J(A) \) (jet ring of \( A \)).
Sweedler theory of the associative operad

Type I adjunctions are quite frequent: if $V = \text{Vect}$

- $E$ finite algebra, $E^* \triangleright -$ is left adjoint to $E \otimes -$,

- $C = k \oplus k\delta$ with $\Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta$
  
  $[C, A] = A[\epsilon]$ and $C \triangleright A = T_A(\Omega_A)$,

- $C = T^c(x)$ (tensor coalgebra)
  

Type II encompasses Sweedler duality: if $V = \text{Vect}$ and $A = k$, we have bijection between

algebra maps $B \rightarrow C^* = [C, k]$

and coalgebra maps $C \rightarrow B^\circ = \{B, k\}$. 
Sweedler theory of the associative operad

Type I adjunctions are quite frequent: if $\mathbf{V} = \mathbf{Vect}$

$E$ finite algebra, $E^* \triangleright -$ is left adjoint to $E \otimes -$,

$\triangleright C = k \oplus k\delta$ with $\Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta$

$[C, A] = A[\epsilon]$ and $C \triangleright A = T_A(\Omega_A),$

$\triangleright C = T^c(x)$ (tensor coalgebra)


Type II encompasses Sweedler duality: if $\mathbf{V} = \mathbf{Vect}$ and $A = k$, we have bijection between

algebra maps $B \rightarrow C^* = [C, k]$

and coalgebra maps $C \rightarrow B^\circ = \{B, k\}$.

Type III encompasses the bar-cobar constructions (if $\mathbf{V} = \mathbf{dgVect}$).
Back to the general theory

The six Sweedler operations of a Hopf operad $P$:

\[
\begin{align*}
\otimes &: P\text{-Coalg} \times P\text{-Coalg} \to P\text{-Coalg} \\
\text{HOM} &: P\text{-Coalg}^{\text{op}} \times P\text{-Coalg} \to P\text{-Coalg} \\
\{-, -\} &: P\text{-Alg}^{\text{op}} \times P\text{-Alg} \to P\text{-Coalg} \\
\triangleright &: P\text{-Coalg} \times P\text{-Alg} \to P\text{-Alg} \\
[-, -] &: P\text{-Coalg}^{\text{op}} \times P\text{-Alg} \to P\text{-Alg} \\
\otimes &: P\text{-Alg} \times P\text{-Alg} \to P\text{-Alg}
\end{align*}
\]
The tensor products are computed termwise (Hadamard).

So is the convolution algebra: for $C$ a $P$-coalgebra and $A$ a $P$-algebra, we have

$$[C, A]_k = [C^k, A_k].$$

This is a $P$-algebra for the product

$$P^k_k \otimes [C, A]_k \longrightarrow P^k_k \otimes P^k_k \otimes [C^k, A_k] \longrightarrow [C^k, C^k] \otimes [C^k, P^k_k \otimes A_k] \longrightarrow [C^k, A_k]$$

A map $C \otimes A \to B$ in $\mathbf{V}^K$ is called a measuring if the corresponding map $A \to [C, B]$ is a $P$-algebra map.
Back to the general theory

For associative algebras $\mu : C \otimes A \to B$ is a measuring iff the following diagram commutes:

\[
\begin{array}{c}
\Delta_{C \otimes A^2} \\
\downarrow C \otimes m_A \\
C \otimes A
\end{array} \quad \overset{\sim}{\longrightarrow} \quad \begin{array}{c}
\mu \otimes \mu \\
B \otimes B
\end{array}
\]
Back to the general theory

\( \mu : C \otimes A \to B \) is a measuring iff the following diagram commutes.

\[
\begin{array}{ccc}
P^k \otimes C^k \otimes A_k & \xrightarrow{\Delta_P} & P^k \otimes P^k \otimes C^k \otimes A_k \\
\downarrow m_A & & \downarrow \Delta_C \\
C^k \otimes P^k \otimes A_k & \cong & P^k \otimes C^k \otimes P^k \otimes A_k \\
\downarrow & & \downarrow \\
P^k \otimes C^k \otimes A_k & \xrightarrow{\mu \otimes n} & P^k \otimes B^k \\
\downarrow m_B & & \downarrow \\
C^k \otimes A_k & \xrightarrow{\mu} & B_k
\end{array}
\]
Back to the general theory

The $P$-algebra $C \triangleright A$ can be defined as the quotient of $P(C \otimes A)$ given by coequalizing the two sides of

$$
\begin{align*}
P_k^k \otimes C_k \otimes A_k & \longrightarrow P_k^k \otimes P_k^k \otimes C_k \otimes A_k \\
\cong & \longrightarrow P_k^k \otimes C_k \otimes P_k^k \otimes A_k \\
\downarrow & \\
C_k^k \otimes P_k^k \otimes A_k & \cong \\
\downarrow & \\
P_k^k \otimes C_k \otimes A_k & \cong \\
\downarrow & \\
P_k^k \otimes P(C \otimes A)_k & \\
\downarrow & \\
P(C \otimes A)_k.
\end{align*}
$$
Sweedler theory of a category $K$

For $P = K$ a category with set of objects $K$, we have

$$P\text{-Alg} = [K, V] \quad \text{and} \quad P\text{-Coalg} = [K^{op}, V].$$

There exists six functors

$\otimes : [K^{op}, V] \times [K^{op}, V] \to [K^{op}, V]$

$\text{Hom} : [K^{op}, V]^{op} \times [K^{op}, V] \to [K^{op}, V]$

$\{−, −\} : [K, V]^{op} \times [K, V] \to [K^{op}, V]$

$\rhd : [K^{op}, V] \times [K, V] \to [K, V]$

$[-, -] : [K^{op}, V]^{op} \times [K, V] \to [K, V]$

$\otimes : [K, V] \times [K, V] \to [K, V]$

By symmetry between $K$ and $K^{op}$ we have

Theorem (?)

1. $[K, V]$ and $[K^{op}, V]$ are symmetric monoidal closed
2. and are enriched, tensored and cotensored over each other.
Sweedler theory of a category $\mathbf{K}$

For $A, B : \mathbf{K} \to \mathbf{V}$ and $C, D : \mathbf{K}^{op} \to \mathbf{V}$ we have:

\[
(C \otimes D)^k = C^k \otimes D^k
\]

\[
\text{Hom}(C, D)^k = \int_{k' \in k/(\mathbf{K}^{op})} [C^{k'}, D^{k'}]
\]

\[
\{A, B\}^k = \int_{k' \in \mathbf{K}/k} [A_{k'}, B_{k'}]
\]

\[
(C \triangleright A)_k = \int_{k' \in \mathbf{K}/k} C^{k'} \otimes A_{k'}
\]

\[
[C, A]_k = [C^k, A_k]
\]

\[
(A \otimes B)_k = A_k \otimes B_k
\]
Sweedler theory of left and right modules over $B$

Let $P = B$ a cocommutative bialgebra, we have

$$P\text{-Alg} = B\text{-Mod} \quad \text{and} \quad P\text{-Coalg} = \text{Mod-B}.$$  

There exists six functors

\begin{align*}
\otimes & : \text{Mod-B} \times \text{Mod-B} \to \text{Mod-B} \\
\text{HOM} & : (\text{Mod-B})^{op} \times \text{Mod-B} \to \text{Mod-B} \\
\{-, -\} & : B\text{-Mod}^{op} \times B\text{-Mod} \to \text{Mod-B} \\
\triangleright & : \text{Mod-B} \times B\text{-Mod} \to B\text{-Mod} \\
[\,-, -\,] & : (\text{Mod-B})^{op} \times B\text{-Mod} \to B\text{-Mod} \\
\otimes & : B\text{-Mod} \times B\text{-Mod} \to B\text{-Mod}
\end{align*}

such that

Theorem

1. $(\text{Mod-B}, \otimes, \text{HOM})$ is symmetric monoidal closed.
2. $(B\text{-Mod}, \{-, -\}, \triangleright, [\,-, -\,], \otimes)$ is enriched, tensored, cotensored and symmetric monoidal over $\text{Mod-B}$. 
Sweedler theory of left and right modules over $B$

For $M$, $N$ two left $B$-modules and $Q$, $R$ two right $B$-modules

$$\text{Hom}(Q, R) = \int_{(B/\ast)^{\text{op}}} [Q, R]$$

$$\{M, N\} = \int_{B/\ast} [M, N]$$

$$(Q \triangleright M) = \int_{B/\ast} Q \otimes M$$

$$[Q, M] = [Q, M]$$

where $B/\ast$ is the division category of the ring $B$

- objects $=$ elements of $B$
- arrows $a \rightarrow b = \text{elements } c \text{ s.t. } a = bc$
Sweedler theory of operads

For $P = OP(K)$ the operad of $K$-colored operads, there exists six functors

\begin{align*}
\otimes & : \text{coOp}(K) \times \text{coOp}(K) \to \text{coOp}(K) \\
\text{Hom} & : \text{coOp}(K)^{\text{op}} \times \text{coOp}(K) \to \text{coOp}(K) \\
\{-, -\} & : \text{Op}(K)^{\text{op}} \times \text{coOp}(K) \to \text{coOp}(K) \\
\triangleright & : \text{coOp}(K) \times \text{Op}(K) \to \text{Op}(K) \\
[-, -] & : \text{coOp}(K)^{\text{op}} \times \text{Op}(K) \to \text{Op}(K) \\
\otimes & : \text{Op}(K) \times \text{Op}(K) \to \text{Op}(K)
\end{align*}

such that

Theorem (A-J)

1. $(\text{coOp}(K), \otimes, \text{Hom})$ is symmetric monoidal closed.
2. $(\text{Op}(K), \{-, -\}, \triangleright, [-, -], \otimes)$ is enriched, tensored, cotensored and symmetric monoidal over $\text{coOp}(K)$. 
Sweedler theory of operads

The monoidal structures are the Hadamard tensor products.

If $C$ is a cooperad and $A$ an operad, $[C, A]$ is the convolution operad of Berger-Moerdijk.

We have formulas

\[
\text{HOM}(C, \text{OP}^\vee(X)) = \text{OP}^\vee([C, X])
\]
\[
\{\text{OP}(X), A\} = \text{OP}^\vee([X, A])
\]
\[
C \triangleright \text{OP}(X) = \text{OP}(C \otimes X)
\]
Part III - MAURER-CARTAN THEORY
Maurer-Cartan theory of algebras

Let \( V = \text{dgVect} \) (= chain complexes),
then \( \text{Alg} = \text{dgAlg} \) and \( \text{Coalg} = \text{dgCoalg} \).
Maurer-Cartan theory of algebras

Let $V = \text{dgVect}$ (chain complexes), then $\text{Alg} = \text{dgAlg}$ and $\text{Coalg} = \text{dgCoalg}$.

For $A$ a dg-algebra, an element $a \in A_{-1}$ is said to be Maurer-Cartan if it satisfies the equation

$$da + a^2 = 0.$$
Maurer-Cartan theory of algebras

Let $\mathbf{V} = \text{dgVect}$ (= chain complexes), then $\text{Alg} = \text{dgAlg}$ and $\text{Coalg} = \text{dgCoalg}$.

For $A$ a dg-algebra, an element $a \in A_{-1}$ is said to be Maurer-Cartan if it satisfies the equation

$$da + a^2 = 0.$$

Let $\text{MC}$ be the dg-algebra generated by a universal Maurer-Cartan element:

$$\text{MC} = k[u]$$

with $|u| = -1$ and $du = -u^2$.

Maurer-Cartan elements of $A$ are in bijection with algebra maps $\text{MC} \to A$. 
Maurer-Cartan theory of algebras

Let $C$ be a dg-coalgebra and $A$ be a dg-algebra.

A twisting cochain from $C$ to $A$ is defined to be a Maurer-Cartan element of the convolution algebra $[C, A]$.

Let $Tw(C, A)$ be the set of twisting cochains from $C$ to $A$. It is in bijection with the set of algebra maps $\text{MC} \rightarrow [C, A]$. 
Maurer-Cartan theory of algebras

The bar construction \( B : \text{dgAlg} \to \text{dgCoalg} \) and the coobar construction \( \Omega : \text{dgCoalg} \to \text{dgAlg} \) are defined to be the functors representing

\[
\text{dgCoalg}^{\text{op}} \times \text{dgAlg} \to \text{Set}
\]

\[
(C, A) \mapsto Tw(C, A)
\]

In other words \( B \) and \( \Omega \) are such that there exists natural bijections between

- twisting cochains \( C \to A \)
- algebra maps \( \Omega C \to A \)
- coalgebra maps \( C \to BA \).
Maurer-Cartan theory of algebras

A twisting cochain is an algebra map \( MC \to [C, A] \).

Using Sweedler operations, we have bijection between the following sets

- algebra maps \( MC \to [C, A] \)
- algebra maps \( C \triangleright MC \to A \)
- coalgebra maps \( C \to \{MC, A\} \).

We deduce that the adjunction of type III

\[
\begin{array}{ccc}
- & \triangleright MC : dgCoalg & \xrightarrow{-} \xleftarrow{\cdot} \xrightarrow{-} dgAlg & : \{MC, -\}
\end{array}
\]

is the bar-cobar adjunction

\[
\Omega : dgCoalg \xleftarrow{-} dgAlg : B
\]

(up to a subtlety about conilpotent coalgebras).
Maurer-Cartan theory of algebras

Recall that $MC = T(u)$ is free as a graded algebra. The formulas

\[
\begin{align*}
\{ T(X), A \} &= T^\vee([X, A]) \\
C \triangleright T(X) &= T(C \otimes X)
\end{align*}
\]

gives the classical construction of the bar and cobar functors

\[
\begin{align*}
BA = \{ MC, A \} &= T^\vee(u^* \otimes A) \\
\Omega C = C \triangleright MC &= T(C \otimes u)
\end{align*}
\]

The internal and external part of the differentials come respectively from the differential of $A$ (or $C$) and of $MC$. 
Operadic Maurer-Cartan theory

Let $P$ be an operad (with one color), the invariant space is

$$Inv(P) = \prod_n P[n]^{\Sigma_n}$$

is a pre-Lie algebra.

A Maurer-Cartan element of $P$ is a Maurer-Cartan element in this pré-Lie algebra.

It is a family of elements $u_n \in P(n)_{-1}$ such that

$$du_n = \sum u_k \circ_i u_{n-k+1}$$
Let $MC$ be the graded operad freely generated by $u_n$ in arity $n$ and degree $-1$ with differential generated by

$$du_n = \sum u_k \circ_i u_{n-k+1}$$

An operad map $MC \to P$ is the same thing as a Maurer-Cartan element of $P$.

We called $MC$ the **Maurer-Cartan operad**.
Operadic Maurer-Cartan theory

An operadic twisting cochain $C \to A$ is a Maurer-Cartan element in the convolution operad $[C, A]$.

The operadic bar and cobar constructions are defined to represent the functor

$$\mathbf{dgCoop}^{op} \times \mathbf{dgOp} \longrightarrow \mathbf{Set}$$

$$(C, A) \longmapsto \text{Tw}(C, A)$$

The Sweedler theory of operads gives us bijections between

- operadic twisting cochains $C \to A$
- operad maps $\Omega C = C \rhd MC \to A$
- cooperads maps $C \to BA = \{MC, A\}$. 
Operadic Maurer-Cartan theory

Recall that $MC = OP(u)$ is free as a graded operad. The formulas

\[
\begin{align*}
\{ OP(X), A \} &= OP^\vee([X, A]) \\
C \triangleright OP(X) &= OP(C \otimes X)
\end{align*}
\]

gives the classical construction of the bar and cobar functors

\[
\begin{align*}
BA &= \{ MC, A \} = OP^\vee(u^* \otimes A) \\
\Omega C &= C \triangleright MC = OP(C \otimes u)
\end{align*}
\]

The internal and external part of the differentials come respectively from the differential of $A$ (or $C$) and of $MC$. 
What is $MC$?
Operadic Maurer-Cartan theory

What is $MC$?

In the symmetric operadic case, an $MC$ algebra structure on $X$ is the same thing as a curved $L_\infty$-algebra structure on $s^{-1}X$.

(In the non-symmetric operadic case, an $MC$ algebra structure on $X$ is the same thing as a curved $A_\infty$-algebra structure on $s^{-1}X$.)

Hence, the curved $L_\infty$ (or $A_\infty$) operads governs the bar and cobar constructions through the Sweedler operation. With a slight abuse of notation:

$$BA = \{cL_\infty, A\} \quad \text{and} \quad \Omega C = C \triangleright cL_\infty.$$
Develop the formalism of Maurer-Cartan for general colored operads.

Apply it to recover all known bar-cobar constructions, including the bar-cobar construction for (co)algebras relative to an operadic twisting cochain.

Understand Koszul complexes and Koszul duality.
Develop the formalism of Maurer-Cartan for general colored operads.

Apply it to recover all known bar-cobar constructions, including the bar-cobar construction for (co)algebras relative to an operadic twisting cochain.

Understand Koszul complexes and Koszul duality.

Thank you.